Functions of Positive and Negative Type, and their Connection with the Theory of Integral Equations.

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(Abstract.)

This memoir is concerned with continuous symmetric functions $\kappa(s, t)$ for which the double integral

$$\int_a^b\int_a^b \kappa(s,t)\,\theta(s)\,\theta(t)\,ds\,dt$$

is either not negative, or not positive, for each function $\theta(s)$ which is continuous in the interval (a, b); in the former case the function $\kappa(s, t)$ is said to be of positive type, while in the latter it is said to be of negative type. The importance of these classes of functions in the theory of integral equations will be gathered from Part I.

The greater portion of the second part is devoted to a proof of the theorem that the necessary and sufficient condition, under which a continuous symmetric function, $\kappa(s, t)$, is of positive type, is that the functions

$$\kappa(s_1, s_1),\ \kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix}, \ldots, \kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix}, \ldots$$

should never be negative, when the variable $s_1, s_2, \ldots, s_n, \ldots$ are each confined to the closed interval (a, b). This leads to several interesting properties of such a function; for instance, if $\kappa(a_1, a_1) = 0$, the functions $\kappa(s, a_1)$, $\kappa(a_1, t)$ are identically zero.

The following parts are devoted to the application of the theorem just quoted in two different directions. Part III contains investigations of certain functions of positive type. In particular, the necessary and sufficient conditions, under which the function

$$\begin{aligned} \kappa(s,t) &= \theta(s)\,\phi(t)\,(s \leq t) \\ &= \phi(s)\,\theta(t)\,(s \geq t) \end{aligned}$$

is of positive type, are obtained. Another theorem proved is to the effect that, when $\kappa(s, t)$ is of positive type, so also are the functions

$$\kappa\begin{pmatrix} s, a_1 \\ t, a_1 \end{pmatrix},\ \kappa\begin{pmatrix} s, a_1, a_2 \\ t, a_1, a_2 \end{pmatrix}, \ldots, \kappa\begin{pmatrix} s, a_1, a_2, \ldots, a_n \\ t, a_1, a_2, \ldots, a_n \end{pmatrix}, \ldots,$$

where $a_1, a_2, \ldots, a_n, \ldots$ are constant numbers lying between, or coinciding with, a and b.

The fourth, and last, part deals with the expansion of functions of positive and negative type in a series of products of normal functions. It is eventually proved that, if $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ are a complete system of normal functions relating to a function $\kappa(s, t)$ of either type, and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are the corresponding singular values, then the series

$$\sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}$$

converges both absolutely and uniformly, and its sum function is $\kappa(s, t)$.

Note on the Spontaneous Luminosity of a Uranium Mineral.

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Rumours of luminosity having been observed in Cornish minerals, in the dark, are not infrequent. I have myself been told of such phenomena by rustics in the mining district, and more than one correspondent has mentioned something similar. None of the accounts have, however, been sufficiently precise to carry conviction, or to afford much clue for further investigation. They have in many cases been obviously exaggerated.

Mr. F. W. Rudler* has quoted a remark by the late Mr. Garby that specimens of uranite "when first discovered by the miners in Huel Buller and Huel Basset were very phosphorescent, so much so that after the lights were extinguished many of the crystals might be discovered *in situ*,"† and he has suggested that this may be in some way connected with the self luminosity of radio-active bodies. The observation would seem to have been made by the miners, not by a scientific observer, and it is implied that the luminosity was of the nature of ordinary phosphorescence, and due to previous exposure to light.

Recently examining a specimen of uranite (autunite), I was struck by its resemblance to the artificially prepared uranium salts, and it occurred to me that in all probability it would be found to exhibit the spontaneous

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* 'Handbook to Minerals of the British Islands,' published by the Geological Survey.

† 'Trans. Roy. Geol. Soc. Cornwall,' vol. 7 (1865), p. 86. Nothing more that is relevant will be found in this.